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Pseudo primitive idempotents and almost 2-homogeneous bipartite distance-regular graphs

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Abstract

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$ and intersection numbers c_i, b_i $(0 \leq i \leq D)$. By a *pseudo cosine sequence* of Γ we mean a sequence of scalars $\sigma_0, \ldots, \sigma_D$ such that $\sigma_0 = 1$ and $c_i\sigma_{i-1} + b_i\sigma_{i+1} = k\sigma_1\sigma_i$ for $1 \leq i \leq D-1$. By an associated *pseudo primitive idempotent* we mean a nonzero scalar multiple of the matrix $\sum_{i=0}^{D} \sigma_i A_i$, where $A_0, \ldots, A_D$ are the distance matrices of Γ. Our main result is the following:

Let $\sigma_0, \ldots, \sigma_D$ denote a pseudo cosine sequence of Γ with $\sigma_1 \notin \{-1, 1\}$ and let E denote an associated pseudo primitive idempotent. The following are equivalent: (i) the entrywise product of E with itself is a linear combination of the all-ones matrix and a pseudo primitive idempotent of Γ; (ii) there exists a scalar β such that $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = 0$ for $1 \leq i \leq D-1$. Moreover, Γ has such a pseudo cosine sequence and pseudo primitive idempotent if and only if Γ is almost 2-homogeneous with $c_2 \geq 2$.

1. Introduction

Let Γ denote a distance-regular graph. Let E and F denote primitive idempotents of Γ. The following question has been considered recently: when is the entrywise product of E and F a linear combination of a “small” number of primitive idempotents of Γ? For work on this and related questions, we refer the reader to the articles of Jurišić, MacLean, Pascasio, Terwilliger and the present author [6–8,10,11].

Terwilliger and Weng [13] defined objects called pseudo primitive idempotents. We give an equivalent definition, first defining associated objects called pseudo cosine sequences. Suppose

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that Γ is bipartite, with diameter D and intersection numbers c_i, b_i $(0 \leq i \leq D)$. By a *pseudo cosine sequence* of Γ we mean a sequence of scalars $\sigma_0, \ldots, \sigma_D$ such that $\sigma_0 = 1$ and $c_i\sigma_{i-1} + b_i\sigma_{i+1} = k\sigma_1\sigma_i$ for $1 \leq i \leq D-1$. (In this article, all scalars are to be taken from the complex numbers unless otherwise specified.) By an associated *pseudo primitive idempotent* we mean a nonzero scalar multiple of the matrix $\sum_{i=0}^{D} \sigma_i A_i$, where $A_0, \ldots, A_D$ are the distance matrices of Γ. It is easy to show that $\sigma_i = 1$ and $\sigma_i = (-1)^i$ are always pseudo cosine sequences. We say that these are *trivial*.

Curtin [5] introduced a combinatorial condition on Γ called the almost 2-homogeneous property. Our main result is the following:

Theorem 1.1. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let $\sigma_0, \ldots, \sigma_D$ denote a nontrivial pseudo cosine sequence of Γ and let E denote an associated pseudo primitive idempotent. The following are equivalent:*

(i) *The entrywise product of E with itself is a linear combination of the all-ones matrix and a pseudo primitive idempotent of Γ.*

(ii) *There exists a scalar β such that $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = 0$ for $1 \leq i \leq D-1$.*

Moreover, Γ has such a pseudo cosine sequence and pseudo primitive idempotent if and only if Γ is almost 2-homogeneous with intersection number $c_2 \geq 2$.

2. Distance-regular graphs

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂, and diameter $D := \max\{\partial(x, y) : x, y \in X\}$. Let k denote a nonnegative integer. We say Γ is *regular* with *valency* k whenever for $x \in X$, $|\{z \in X : \partial(x, z) = 1\}| = k$. We say Γ is *distance-regular* whenever for $0 \leq h, i, j \leq D$ and $x, y \in X$ with $\partial(x, y) = h$, the scalar $p^h_{ij} := |\{z \in X : \partial(x, z) = i, \partial(y, z) = j\}|$ is independent of x and y. For notational convenience, set $c_i := p^i_{1i-1} (1 \leq i \leq D)$, $a_i := p^i_{1i} (0 \leq i \leq D)$, $b_i := p^i_{1i+1} (0 \leq i \leq D-1)$, $c_0 := 0$ and $b_D := 0$.

Suppose that Γ is distance-regular. We observe that Γ is regular with valency $k = b_0$. Further, we observe that

$$c_i + a_i + b_i = k \quad (0 \leq i \leq D). \tag{1}$$

We recall that Γ is *bipartite* whenever $a_i = 0$ for $0 \leq i \leq D$.

3. The almost 2-homogeneous property

For the rest of this article, let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. In this section, we recall the definitions of 2-homogeneity and almost 2-homogeneity and collect some useful results pertaining to the latter property.

Definition 3.1 (*[9]*). Γ is said to be *2-homogeneous* whenever for $1 \leq i \leq D-1$ and $x, y, z \in X$ with $\partial(x, y) = 2$, $\partial(x, z) = i$ and $\partial(y, z) = i$, the quantity $|\{w \in X : \partial(x, w) = 1, \partial(y, w) = 1, \partial(z, w) = i-1\}|$ is independent of x, y and z.

Definition 3.2 (*[5]*). Γ is said to be *almost 2-homogeneous* whenever for $1 \leq i \leq D-2$ and $x, y, z \in X$ with $\partial(x, y) = 2$, $\partial(x, z) = i$ and $\partial(y, z) = i$, the quantity $|\{w \in X : \partial(x, w) = 1, \partial(y, w) = 1, \partial(z, w) = i-1\}|$ is independent of x, y and z.

Curtin characterizes almost 2-homogeneous graphs in three cases based on the value of c_2.

Lemma 3.3 (*[5, Theorem 4.4],[3, Theorem 6.5.1]*). *The following are equivalent:*

(i) *Γ is almost 2-homogeneous with $c_2 = 1$.*
(ii) *$c_i = 1$ for $1 \le i \le D - 1$.*

Suppose that (i) *and* (ii) *hold. Then Γ is a generalized $2D$-gon of order $(1, k-1)$ and $D \in \{4, 6\}$. Γ is not 2-homogeneous.*

Lemma 3.4 (*[5, Theorem 4.7],[2]*). *The following are equivalent:*

(i) *Γ is almost 2-homogeneous with $c_2 = 2$.*
(ii) *$c_i = i$ for $1 \le i \le D - 1$.*

Suppose that (i) *and* (ii) *hold. Then we have one of the following: $c_D = D$ and Γ is the D-cube; $c_D = 2D$ and Γ is the folded $2D$-cube; or $D = 4$, $c_D = 24$ and Γ is the coset graph of the extended binary Golay code. Γ is 2-homogeneous if and only if Γ is the D-cube.*

Lemma 3.5 (*[5, Theorems 4.8, 4.9]*). *The following are equivalent:*

(i) *Γ is almost 2-homogeneous with $c_2 \ge 3$.*
(ii) *There exist scalars q and m such that*

$$c_i = \frac{(q^{2i} - 1)(mq^2 - 1)}{(q^2 - 1)(mq^{2i} - 1)} \quad (0 \le i \le D - 1), \tag{2}$$

$$b_i = \frac{(m^2 q^{2i} - 1)(mq^2 - 1)}{m(q^2 - 1)(mq^{2i} - 1)} \quad (0 \le i \le D - 1), \tag{3}$$

with all denominators nonzero.

Suppose that (i) *and* (ii) *hold. Then $D \in \{4, 5\}$. Γ is 2-homogeneous if and only if $mq^D = -1$.*

4. Pseudo cosine sequences and idempotents

After reviewing the definitions of cosine sequences and primitive idempotents, we define pseudo cosine sequences and pseudo primitive idempotents and give some basic results about them. We note that our definition of pseudo primitive idempotent is equivalent to the original one in [13].

We recall the Bose–Mesner algebra. Let $\mathbb{C}$ denote the field of complex numbers. By $\mathrm{Mat}_X(\mathbb{C})$ we mean the $\mathbb{C}$-algebra consisting of all matrices whose entries are in $\mathbb{C}$ and whose rows and columns are indexed by X. For $0 \le i \le D$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{otherwise} \end{cases} \quad (x, y \in X).$$

We call $A_0, \ldots, A_D$ the *distance matrices* of Γ. Note that the entrywise product satisfies

$$A_i \circ A_j = \delta_{ij} A_i \quad (0 \le i, j \le D). \tag{4}$$

Abbreviate $A := A_1$. We call A the *adjacency matrix* of Γ. Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A. By [1, Theorem 20.7], the matrices $A_0, \ldots, A_D$ form a basis for M. We call M the *Bose–Mesner algebra* of Γ.

Next we recall primitive idempotents and cosine sequences. By [3, Theorem 2.6.1], M has a basis $E_0, \ldots, E_D$ such that $E_i E_j = \delta_{ij} E_i$ for $0 \le i, j \le D$. We call $E_0, \ldots, E_D$ the *primitive idempotents* of Γ. Observe that there exists a sequence of scalars $\theta_0, \ldots, \theta_D$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. We call θ_i the *eigenvalue* of Γ associated with E_i $(0 \le i \le D)$.

Let θ denote an eigenvalue of Γ and let E denote the associated primitive idempotent. Since the distance matrices form a basis for M, there exist scalars $\sigma_0, \ldots, \sigma_D$ such that $E = m|X|^{-1} \sum_{i=0}^{D} \sigma_i A_i$, where m is the rank of E. We call $\sigma_0, \ldots, \sigma_D$ the *cosine sequence* associated with θ and E. By [3, p. 128], $\sigma_0 = 1$ and $c_i \sigma_{i-1} + b_i \sigma_{i+1} = \theta \sigma_i$ for $0 \le i \le D$, where σ_{-1} and σ_{D+1} denote indeterminates.

Now we define pseudo cosine sequences and pseudo primitive idempotents, generalizations of cosine sequences and primitive idempotents.

Definition 4.1. Consider a scalar θ. By the associated *pseudo cosine sequence* we mean the sequence of scalars $\sigma_0, \ldots, \sigma_D$ such that $\sigma_0 = 1$ and

$$c_i \sigma_{i-1} + b_i \sigma_{i+1} = \theta \sigma_i \quad (0 \le i \le D - 1), \tag{5}$$

where σ_{-1} denotes an indeterminate. Suppose that $\sigma_0, \ldots, \sigma_D$ is the pseudo cosine sequence associated with θ. By a *pseudo primitive idempotent* associated with θ and $\sigma_0, \ldots, \sigma_D$ we mean a nonzero scalar multiple of the matrix

$$\sum_{i=0}^{D} \sigma_i A_i . \tag{6}$$

We say that this matrix itself is the *canonical* associated pseudo primitive idempotent.

Note that setting $i = 0$ in (5) gives $\theta = k\sigma_1$. We now mention a basic example of a pseudo primitive idempotent and pseudo cosine sequence and a lemma following immediately from Definition 4.1. Motivated thereby, we then give another definition.

Example 4.2. Let J denote the all-ones matrix in $\mathrm{Mat}_X(\mathbb{C})$. Then J is a pseudo primitive idempotent of Γ, associated with the valency k and with pseudo cosine sequence $1, 1, \ldots, 1$.

Lemma 4.3. *Given a scalar θ, let $\sigma_0, \ldots, \sigma_D$ denote the associated pseudo cosine sequence of Γ. The pseudo cosine sequence associated with $-\theta$ is $\sigma_0, -\sigma_1, \sigma_2, \ldots, (-1)^D \sigma_D$.*

Definition 4.4. Given a scalar θ, let E and $\sigma_0, \ldots, \sigma_D$ denote the associated pseudo primitive idempotent and pseudo cosine sequence of Γ. We say E (or $\sigma_0, \ldots, \sigma_D$) is *trivial* whenever $\theta \in \{-k, k\}$ (i.e., $\sigma_1 \in \{-1, 1\}$).

We list some useful facts about pseudo cosine sequences.

Lemma 4.5. *Given a scalar θ, let $\sigma_0, \ldots, \sigma_D$ denote the associated pseudo cosine sequence of Γ. Then*

$$\sigma_0 = 1, \qquad \sigma_1 = \frac{\theta}{k}, \qquad \sigma_2 = \frac{\theta^2 - k}{k(k-1)}, \qquad \sigma_3 = \frac{\theta(\theta^2 - k - c_2(k-1))}{k(k-1)(k-c_2)}. \tag{7}$$

Proof. This follows from repeated application of (5), together with (1). $\square$

Lemma 4.6. *Let $\sigma_0, \ldots, \sigma_D$ denote a pseudo cosine sequence of Γ.*

(i) *For* $1 \leq i \leq D-1$,

$$c_i(\sigma_{i-1} - \sigma_{i+1}) = k(\sigma_1\sigma_i - \sigma_{i+1}) \quad \text{and} \quad b_i(\sigma_{i+1} - \sigma_{i-1}) = k(\sigma_1\sigma_i - \sigma_{i-1}).$$

(ii) *Suppose that* $\sigma_{i-1} \neq \sigma_{i+1}$ *for some* i $(1 \leq i \leq D-1)$. *Then*

$$c_i = k\frac{\sigma_1\sigma_i - \sigma_{i+1}}{\sigma_{i-1} - \sigma_{i+1}}, \tag{8}$$

$$b_i = k\frac{\sigma_1\sigma_i - \sigma_{i-1}}{\sigma_{i+1} - \sigma_{i-1}}.$$

In particular, $\sigma_1\sigma_i - \sigma_{i+1}$ *and* $\sigma_1\sigma_i - \sigma_{i-1}$ *are nonzero.*

(iii) *Suppose that* $\sigma_0 \neq \sigma_2$. *Then*

$$k = \frac{1-\sigma_2}{\sigma_1^2 - \sigma_2}.$$

Proof. Part (i) follows from Definition 4.1 in view of (1). Part (ii) is then immediate. Setting $i = 1$ in (8) and solving for k produces part (iii). $\square$

Lemma 4.7. *Let* $\sigma_0, \ldots, \sigma_D$ *denote a pseudo cosine sequence. For* $1 \leq i \leq D-1$, *if* $\sigma_i = 0$ *then* $\sigma_{i+1} \neq 0$.

Proof. Suppose that $\sigma_i = \sigma_{i+1} = 0$. By (5) and since $c_i \neq 0$, we see $\sigma_{i-1} = 0$. Iterating this argument, we find $\sigma_0 = 0$, a contradiction. $\square$

5. Results

We consider conditions on pseudo cosine sequences and pseudo primitive idempotents and how these conditions relate to each other and to almost 2-homogeneity.

Definition 5.1. Given a scalar θ, let E and $\sigma_0, \ldots, \sigma_D$ denote the associated pseudo primitive idempotent and pseudo cosine sequence of Γ. Let β and γ denote scalars. Following [12], we say θ (or E or $\sigma_0, \ldots, \sigma_D$) is (β, γ)-*recurrent* whenever

$$\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = \gamma \tag{9}$$

for $1 \leq i \leq D-1$. In this article, we are primarily concerned with $(\beta, 0)$-recurrence. We say simply that an object is *recurrent* whenever there exists a β such that the object is $(\beta, 0)$-recurrent.

Lemma 5.2. *Let* Φ *denote the set of all recurrent scalars. Then one of the following holds:*

(i) $\Phi = \{-k, k\}$.

(ii) $c_2 \geq 2$ *and* $\Phi = \{-k, -\theta, \theta, k\}$, *where*

$$\theta^2 = \frac{(k-2)(k-c_2)}{c_2 - 1} \tag{10}$$

is nonzero.

Proof. That k is $(2, 0)$-recurrent follows from Example 4.2. That a scalar θ is $(\beta, 0)$-recurrent if and only if $-\theta$ is $(-\beta, 0)$-recurrent follows from Lemma 4.3.

Let θ denote a scalar other than k and $-k$ and suppose that θ is $(\beta, 0)$-recurrent. Let $\sigma_0, \ldots, \sigma_D$ denote the pseudo cosine sequence associated with θ.

We claim $\theta \neq 0$. Suppose that $\theta = 0$. Setting $i = 1$ in (9) and eliminating the σ_i's using (7), we see $k \in \{0, 2\}$, a contradiction.

Now we show (10). Since θ is nonzero, σ_1 is also nonzero. Thus when we set $i = 1$ in (9) we find $\beta = (\sigma_0 + \sigma_2)/\sigma_1$. Using this when we set $i = 2$ in (9) and using (7) to eliminate the σ_i's from the result, we find

$$0 = \sigma_1 - \beta\sigma_2 + \sigma_3 = \frac{(\theta^2 - k^2)((c_2 - 1)\theta^2 - (k-2)(k-c_2))}{k(k-1)^2(k-c_2)\theta}.$$

The first factor in the numerator is nonzero by assumption, so the second factor must be zero. Since $k > 2$ and $k > c_2$, we see $c_2 > 1$ and we have (10). □

We are interested in the entrywise product of a pseudo primitive idempotent with itself. The following lemma will be helpful.

Lemma 5.3. *Given a scalar θ, let E denote the associated canonical pseudo primitive idempotent. Let E' denote the canonical pseudo primitive idempotent associated with $-\theta$. Then $E \circ E = E' \circ E'$.*

Proof. Let $\sigma_0, \ldots, \sigma_D$ denote the pseudo cosine sequence associated with E. By (6) and Lemma 4.3, $E' = \sum_{i=0}^{D} (-1)^i \sigma_i A_i$. So by (4), we see $E' \circ E' = \sum_{i=0}^{D} \sigma_i^2 A_i = E \circ E$. □

We now re-examine almost 2-homogeneity, dealing with each of Curtin's cases in turn. In each case, we investigate the existence of nontrivial recurrent pseudo cosine sequences.

Lemma 5.4. *Suppose that Γ is almost 2-homogeneous with $c_2 = 1$. Then Γ does not have a nontrivial recurrent pseudo cosine sequence.*

Proof. Immediate from Lemma 5.2. □

Lemma 5.5. *Suppose that Γ is almost 2-homogeneous with $c_2 = 2$. Then*

$$\sigma_i = 1 - \frac{2}{k} i \quad (0 \leq i \leq D)$$

and

$$\tau_i = 1 - \frac{4}{k(k-1)} i(k-i) \quad (0 \leq i \leq D)$$

are pseudo cosine sequences of Γ, associated with the scalars $k - 2$ and $k - 4$, respectively. Let E and F be the canonical pseudo primitive idempotents associated with these pseudo cosine sequences. Then $E \circ E = xJ + (1-x)F$, where $x = k^{-1}$. We note that E is nontrivial. Moreover, E is $(2, 0)$-recurrent and F is $(2, \gamma)$-recurrent, where $\gamma = 8k^{-1}(k-1)^{-1}$.

Proof. We verify that the σ_i and τ_i formulae give pseudo cosine sequences using Lemma 3.4 and (5). We check the assertion about the pseudo primitive idempotents by using (4) and (6) to expand each side of the equation as a linear combination of the distance matrices and then comparing coefficients. The recurrence statements are routine. □

Lemma 5.6. *Suppose that Γ is almost 2-homogeneous with $c_2 \geq 3$. Let m and q be as in Lemma 3.5. Then*

$$\sigma_i = 1 + \frac{(q^i - 1)(mq^i - 1)}{(m+1)q^i} \quad (0 \leq i \leq D)$$

and

$$\tau_i = 1 + \frac{(q^{2i} - 1)(m^2q^{2i} - 1)(mq^2 - 1)}{(m^2q^2 - 1)(m+1)q^{2i}} \quad (0 \leq i \leq D)$$

are pseudo cosine sequences of Γ, associated with the scalars

$$\frac{(1 - mq^2)(1 + mq^2)}{mq(1 - q^2)} \quad \text{and} \quad \frac{(1 - mq^2)(1 + mq^4)}{mq^2(1 - q^2)},$$

respectively. Let E and F be the canonical pseudo primitive idempotents associated with these pseudo cosine sequences. Then $E \circ E = xJ + (1-x)F$, where $x = m(q^2 - 1)(m+1)^{-1}(mq^2 - 1)^{-1}$. We note that E is nontrivial. Moreover, E is $(q + q^{-1}, 0)$-recurrent and F is $(q^2 + q^{-2}, \gamma)$-recurrent, where $\gamma = -m(m-1)(q^4 - 1)(q^2 - 1)(m+1)^{-1}(m^2q^2 - 1)^{-1}q^{-2}$.

Proof. Similar to the proof of Lemma 5.5, but using Lemma 3.5 instead of Lemma 3.4. Note that the denominators are all nonzero, since otherwise we would have contradictory values in (2) and (3): $m = -1$ makes $k = 0$, $q = 0$ makes $c_2 = 1$, $m^2q^2 = 1$ makes $b_1 = 0$. The restrictions on m and q also show that $\sigma_1 \notin \{-1, 1\}$, so E is nontrivial. □

Next we consider the pseudo primitive idempotent behavior described in the previous two lemmas, showing that it implies recurrence. We then tie that back to almost 2-homogeneity.

Lemma 5.7. *Let E and F denote pseudo primitive idempotents of Γ and suppose that E is nontrivial. Suppose that $E \circ E$ is a linear combination of J and F. Then E is recurrent.*

Proof. Without loss of generality, suppose that E and F are canonical. Let $\sigma_0, \ldots, \sigma_D$ (resp. $\tau_0, \ldots, \tau_D$) denote the pseudo cosine sequence associated with E (resp. F).

Apparently, there exist scalars x and y such that $E \circ E = xJ + yF$. Using (6) and Example 4.2 to eliminate E, F and J from this equation, applying (4) and equating coefficients of A_i in the result, we find

$$\sigma_i^2 = x + y\tau_i \quad (0 \leq i \leq D). \tag{11}$$

Setting $i = 0$ in (11), we see $x + y = 1$. We claim $y \neq 0$. On the contrary, suppose that $y = 0$. Then $x = 1$. Setting $i = 1$ in (11), we see $\sigma_1 \in \{-1, 1\}$, which contradicts the assumption that E is nontrivial. So $y \neq 0$. Now we can eliminate y from (11) and solve the result to get

$$\tau_i = \frac{\sigma_i^2 - x}{1 - x} \quad (0 \leq i \leq D). \tag{12}$$

Using Lemma 4.6(i), we find that the σ_i obey

$$\frac{c_i}{k}(\sigma_{i-1} - \sigma_{i+1}) = \sigma_1\sigma_i - \sigma_{i+1} \quad (1 \leq i \leq D-1) \tag{13}$$

and the τ_i obey a similar equation. Using (12) to eliminate τ_{i-1}, τ_i and τ_{i+1} from the latter and simplifying the result, we get

$$\frac{c_i}{k}(\sigma_{i-1} - \sigma_{i+1})(\sigma_{i-1} + \sigma_{i+1}) = \tau_1(\sigma_i^2 - x) - (\sigma_{i+1}^2 - x) \quad (1 \leq i \leq D-1). \tag{14}$$

Subtracting (14) from $(\sigma_{i-1}+\sigma_{i+1})$ times (13) and using (12) with $i=1$ to eliminate x from the result, we get

$$\sigma_1\sigma_i\sigma_{i-1}-\sigma_{i-1}\sigma_{i+1}+\sigma_1\sigma_i\sigma_{i+1}=\tau_1(\sigma_i^2-1)+\sigma_1^2 \quad (1\le i\le D-1). \tag{15}$$

Setting $i=1$ in (15) and simplifying, we find $\tau_1=\sigma_2$.

We claim $\sigma_1\neq 0$. Suppose the contrary. Setting $i=2$ in (15), we find $\sigma_2(\sigma_2^2-1)=0$. But $\sigma_2=-1/(k-1)$ by (7), so this is a contradiction.

Our goal is to show that there exists a scalar β such that $\sigma_{i-1}-\beta\sigma_i+\sigma_{i+1}=0$ for $1\le i\le D-1$. For brevity, let $p(i)$ denote the left side of this equation. Since $\sigma_1\neq 0$, we can choose β such that $p(1)=0$. Note that $\tau_1(=\sigma_2)=\beta\sigma_1-1$. Replacing i by $i+1$ in (15), subtracting (15) and rearranging terms, we find

$$(\sigma_1\sigma_i-\sigma_{i+1})p(i)=(\sigma_1\sigma_{i+1}-\sigma_i)p(i+1) \quad (1\le i\le D-2). \tag{16}$$

If we can show that $\sigma_1\sigma_{i+1}-\sigma_i$ is nonzero for $1\le i\le D-2$ then we are done by (16), the fact that $p(1)=0$, and induction.

Suppose the contrary and let j denote the smallest integer such that $\sigma_1\sigma_{j+1}-\sigma_j=0$. So the induction only proceeds to $p(j)=0$. From Lemma 4.6(i) we see that $\sigma_j=\sigma_{j+2}=\sigma_1\sigma_{j+1}$. Observe that $\sigma_{j+1}\neq 0$ since otherwise $\sigma_j=0$ as well, contradicting Lemma 4.7.

We claim $\sigma_1\sigma_{j+2}-\sigma_{j+1}$ is nonzero. Suppose the contrary. Then $\sigma_1^2\sigma_{j+1}-\sigma_{j+1}$ is zero. But neither σ_1^2-1 nor σ_{j+1} is zero, so this is impossible. Thus $\sigma_1\sigma_{j+2}-\sigma_{j+1}$ is nonzero.

Combining this, (16) with $i=j+1$, and the fact that $\sigma_1\sigma_{j+1}-\sigma_{j+2}=0$, we see $p(j+2)=0$. From the equation $p(j)=0$ (resp. $p(j+2)=0$), we find $\sigma_{j-1}=\beta\sigma_j-\sigma_{j+1}$ (resp. $\sigma_{j+3}=\beta\sigma_{j+2}-\sigma_{j+1}$).

Now, σ_{j+2} and σ_j each equal $\sigma_1\sigma_{j+1}$. Also, σ_{j+3} and σ_{j-1} each equal $(\beta\sigma_1-1)\sigma_{j+1}$. So by Lemma 4.6, b_j and c_{j+2} each equal 1. But $1\le c_j\le c_{j+2}$ [3, Proposition 4.1.6(i)], so $c_j=1$ as well. Thus $k=c_j+b_j=2$, a contradiction. □

Lemma 5.8. *If Γ has a nontrivial recurrent pseudo cosine sequence then Γ is almost 2-homogeneous.*

Proof. Let $\sigma_0,\dots,\sigma_D$ denote a nontrivial $(\beta,0)$-recurrent pseudo cosine sequence of Γ. We solve (9) for the σ_i's. Our treatment starts with application of the theory of recurrence relations, which depends on the value of β.

If $\beta=2$ (resp. $\beta=-2$) then there exist scalars a and b such that $\sigma_i=a+bi$ (resp. $\sigma_i=(a+bi)(-1)^i$) for $0\le i\le D$. Since $\sigma_0=1$, we have $a=1$. By nontriviality, $b\neq 0$. Now we use Lemma 4.6 to show that $c_i=i$ for $0\le i\le D-1$. By Lemma 3.4, Γ is almost 2-homogeneous.

Suppose that $\beta\notin\{-2,2\}$. Then there exist scalars a, b and q such that $\sigma_i=aq^i+bq^{-i}$. Clearly, $q\neq 0$. Since $\sigma_0=1$, we see $b=1-a$. By nontriviality and Lemma 4.5, $\sigma_2\neq\sigma_0$. Thus $q^2\neq 1$. By Lemma 4.6(ii) then $\sigma_1^2\neq\sigma_2$. Thus $a\notin\{0,1\}$. Set $m:=a/(1-a)$ and note that $m\neq 0$. Note also that $m\neq -1$, which allows us to write $a=m/(1+m)$. Suppose that $m=1$, making $a=1/2$. Then $k=2$ by Lemma 4.6(iii). This is a contradiction, so $m\neq 1$. Given $1\le i\le D-1$, we observe some equivalences that follow from our formula for σ_i and Lemma 4.6(i): $mq^{2i}=1$ if and only if $\sigma_{i-1}=\sigma_{i+1}$ if and only if $\sigma_1\sigma_i=\sigma_{i+1}$ if and only if $q^{2i}=1$. Since $m\neq 1$, none of these hold. Writing σ_i in terms of q and m, we can thus use Lemma 4.6(ii) to find that the intersection numbers of Γ are as in (2) and (3), with all denominators nonzero. Now by Lemma 3.5, Γ is almost 2-homogeneous. □

Putting our results together, we have the following:

Proof of Theorem 1.1. First suppose that (i) holds. Then (ii) holds by Lemma 5.7.

Next suppose that (ii) holds. Then Γ is almost 2-homogeneous by Lemma 5.8. Now $c_2 \geq 2$ by Lemma 5.2.

Finally, suppose that Γ is almost 2-homogeneous with $c_2 \geq 2$. By Lemmas 5.5 and 5.6, there is a nontrivial pseudo primitive idempotent whose entrywise product with itself is a linear combination of J and a pseudo primitive idempotent of Γ. Even if E is not this nontrivial pseudo primitive idempotent, (i) holds for E by Lemmas 5.2 and 5.3. □

We conclude with some details and comments on Theorem 1.1.

Theorem 5.9. *Referring to Theorem 1.1, suppose that* (i)–(ii) *hold. Let x and y be scalars and let F be a pseudo primitive idempotent of Γ such that $E \circ E = xJ + yF$. Suppose that E and F are the canonical pseudo primitive idempotents associated with the scalars θ and ϕ, respectively.*

(i) $x = k^{-1}$ *and* $y = 1 - k^{-1}$.
(ii) $\theta^2 = \frac{(k-2)(k-c_2)}{c_2-1}$ *and* $\phi = \frac{k-2c_2}{c_2-1}$.
(iii) $\phi \notin \{-k, k, -\theta, \theta\}$.
(iv) $\beta = \frac{c_2(k-2)}{\theta}$.
(v) *For some scalar γ, the pseudo cosine sequence associated with F is $(\beta^2 - 2, \gamma)$-recurrent.*

Proof. These follow from Lemmas 3.5, 5.2, 5.5 and 5.6. □

Theorem 5.10. *Referring to Theorem 1.1, suppose that* (i)–(ii) *hold. The following are equivalent:*

(i) *Γ is 2-homogeneous.*
(ii) *$\sigma_0, \ldots, \sigma_D$ is a cosine sequence of Γ.*

Proof. This follows from [4, Lemma 27]. □

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